STOCHASTIC DIFFERENTIAL EQUATION MODELING FOR ELECTRICAL NETWORKS

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Abstract

In this paper deterministic second order ordinary differential equation and its stochastic analogue is presented. This describes the concentration of charge in the capacitor of a simple L-C-R circuit and can be solved explicitly in both the deterministic and stochastic cases.

Keywords

Stochastic differential equations, Markov diffusion process, Fourier transform, Non-transient solution, Numerical Scheme.

1. Introduction

During the past fifty years, interest in the study of stochastic phenomena has increased dramatically. Intensified research activity in this area has been stimulated by the need to take into account random effects in complicated physical systems. For such systems, Markov processes have provided a natural replacement for deterministic functions as mathematical descriptions of state. Consequently, a number of methods for constructing these processes have been introduced. One such method that is well known and extensively applied obtains the distribution of the process from Kolmogorovs equations; however these partial differential equations for the transition density of the process can be difficult to solve.

Another technique of particular relevance when the random effects can be considered as external to some reasonably well understood deterministic system replaces input parameters in the deterministic mode by random processes; the resulting random model characterizes the sample path structure of the solution process.

An example that has had wide application in engineering consists of adding a noise term to the right side of a deterministic equation. Langevin formulated the first such model in 1908 to describe the velocity of a particle moving in a random force field. Random differential equations of this type can be interpreted as stochastic differential equations, following Ito's basic work in the early 1940s. Solutions of such equations represent Markov diffusion processes, the prototype of which is the Brownian motion process alternatively called Wiener process. The theory of stochastic differential equation set down by Ito[2], and independently established in the Soviet Union by Gikhman, together with the previous mathematical work of Wiener and Levy on Brownian motion has provided the basic tools makings this more ambition approach of constructing sample paths feasible.

In this paper, the properties of the explicit solution (when possible) of the deterministic and stochastic versions of second-order ordinary differential equation is explored. The equations presented here, model the concentration of charge at a fixed point of a simple L-C-R series circuit in the presence of deterministic and stochastic electromotive source.

2. Deterministic and Stochastic Differential A. Equation Modelling for Simple Electrical Networks

Let Q(t) be the charge on the capacitor of a simple L-C-R series circuit. Let also F(t) denote the electromotive source provided to the system at time t(this may be a battery or a generator which produces a potential difference and causes a current I to flow through the circuit). R represents a resistance to the flow of the current such as that produced by a light bulb. When current flows through a coil or wire L, a magnetic field is produced which opposes any change in the current through the coil. The change in voltage produced by the coil is proportional to the rate of change of the current and the constant of proportionality is called the inductance L of the coil. A capacitor, or condenser, indicated by C, usually consists of two metal plates separated by a material through which very little current can flow; the flow of the current is reversed as one plate or the other becomes fully charged.

To derive a differential equation, which is satisfied by Q(t), one may use the Kirchhoff's second law which states that the impressed voltage in closed circuit equals the sum of the voltage drops in the rest of the circuit. By Ohm's Law the voltage drop across a resistance equals R*I. The voltage drop across the inductance of L henrys equals L(dI/ dt) and across the capacitance of C farads equals Q/C. Hence

$$F(t) = L\frac{dI}{dt} + R*I + \frac{Q}{C}$$

and since $I(t) = \frac{dQ(t)}{dt}$, the following equation holds

$$L.Q''(t) + R.Q'(t) + \frac{1}{C} Q(t) = F(t) \text{ ------- } (1.1)$$

For the derivation of the solution of this second order ordinary differential equation a number of methods may be applied depending on the type of the electromotive source function. When F(t) is piecewise continuous with a finite number of maximums and minimums and

$$\int_{-\infty}^{\infty} |F(t)|^2 dt < \infty$$

holds, Fourier transforms may be applied.

The Fourier transforms of F(t) is

$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(t)dt.$$

Suppose also that

$$\tilde{Q}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q_s(t)e^{i\omega t} dt.$$

is the Fourier transform of the specific solution of the non-homogeneous differential equation. Then it holds that

$$\{ L(-i\omega)^2 + R(-i\omega) + C^{-1} \} \tilde{Q}_s(\omega) = -i\omega \tilde{F}(\omega)$$

$$\tilde{Q}_s(\omega) = \frac{-i\omega}{-\omega^2 L - i\omega R\_C^{-1}} \qquad \tilde{F}(\omega) = \frac{\tilde{F}(\omega)}{Z(\omega)}$$

Where $Z(\omega) = R - i\left(\omega L - \frac{1}{\omega C}\right)$

is called the impedance of the circuit. Hence

$$Q_s(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{F}(\omega)}{Z(\omega)} e^{-i\omega t} d\omega$$

is the non-transient solution.

B. Genesis Scheme and Solution for the Stochastic Case

The potential source may not be deterministic but of the form

F(t) = G(t) + "noise"

As proposed by Oksendal [7]. The equation obtained by allowing randomness in the coefficients of a differential equation is a stochastic differential equation. Here, the case where the noise is described by a Wiener process is discussed. Equation (1.1) is being modified to give

$$LQ_t'' RQ_t' + C^{-1}Q_t = G_t + aW_t \text{ ---------- } (1.2)$$

Where a is called the intensity of the noise and its magnitude determines the deviation of the stochastic case from the deterministic one. By introducing the vector.

$$X = X(t, \omega) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} Q_t \\ Q_{t'} \end{pmatrix}$$

(1.2) may be transformed to give

$$X_1' = X_2$$

$$L X_2' = -RX_2 - C^{-1} X_1 + G_t + aW_t$$

or in matrix notation

$$dX = AXdt + Hdt + KdW_t \text{ ---------- } (1.3)$$

where

$$dX = \begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -(CL)^{-1} & -RL^{-1} \end{pmatrix}, Ht = \begin{pmatrix} 0 \\ L^{-1} G_t \end{pmatrix}, K = \begin{pmatrix} 0 \\ L^{-1} \end{pmatrix}$$

and W_t is a one-dimensional Wiener process. Equation (1.3) represents a 2-diamensional stochastic differential equation and may be written as

$$\exp(-At)dX - \exp(-At)AXdt = \exp(-At)[H_t dt + KdW_t] \text{ ----(1.4)}$$

where for a general nxn matrix F,

$$\exp(F) = \sum_{n=0}^{\infty} \frac{F^n}{n!}$$

the series converge for every F and its summation has the properties

$$(e^{As} e^{At}) = e^{A(s+t)}, (e^{At})(e^{-At}) = I, \frac{d}{ds}(e^{As}) = Ae^{At}$$

When F can be written as $F = SAS^{-1}$ the differential equation

$$\frac{du}{dt} = Fu \quad \text{may be solved to give}$$

$$u(t) = e^{Ft} u_0 = Se^{At} S^{-1} u_0$$

The columns of S are eigenvectors of F so

$$u(t) = [x_1, x_2, \ldots\ldots x_n] \begin{pmatrix} e^{\lambda 1t} & \ldots\ldots & 0 \\ 0 & \ldots\ldots & 0 \\ 0 & \ldots\ldots & e^{\lambda nt} \end{pmatrix}$$

$$S^{-1} u_0 = c_1 e^{\lambda 1t} x_1 + \ldots\ldots\ldots\ldots + c_n e^{\lambda nt} x_n$$

The general solution is a combination of exponential functions and the constants c_t are given by the relation $c = S^{-1}u_0$ here it is tempting to relate the left hand side of (1.4) to d(exp(-At) X). To do this we use the 2-dimensional It's formula (for a thorough discussion on Ito's formula see Oksendsal [7]). Applying this result to the two coordinate functions g_1, g_2 of $g : [0,\infty) \times R^2 \rightarrow R^2$ given by

$$g(t,x_1,x_2)=\exp(-At)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We obtain that

$$d(\exp(-At)) = (-At)\exp(-At)Xdt + \exp(-At)dX.$$

Substituted in (1.4) this gives

$$\exp(-At)X-X_0 = \int_0^t \exp(-As)H_s ds + \int_0^t \exp(-As)\, KdW_s,$$

Where the last integral is an Ito integral (see Ikeda and Watanabe). Now, the integration by parts theorem (Oksendal [7]) may be applied to give

$$X=\exp(At)[X_0+\exp(-At)KW_t+\int_0^t \exp(-As)[H_s-AKW_s]ds]$$

3. Conclusion

Nowadays, stochastic dynamical systems take the place of the deterministic ones, when modeling of physical or mechanical processes is of concern. As expected through, the derivation of the explicit solutions (when possible) or the construction of efficient numerical schemes for the approximation of the trajectories, are more complex procedures compared to the ones of the deterministic cases. The example presented in this paper supports this argument.

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Biography

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